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Skolem Functions for false QBFs

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SAT

- Propositional formulas (in CNF)

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$$F(x_1, x_2, y_1) = (x_1 \lor \neg y_1) \land (x_2 \lor \neg y_1)$$

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$$F(x_1, x_2, y_1) = (x_1 \vee \neg y_1) \wedge (x_2 \vee \neg y_1)$$

- Result: sat or unsat

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$$x_1 = \bot, x_2 = \bot, y_1 = \bot$$

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$$x_1 = \top, x_2 = \top, y_1 = \top$$

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$$x_1 = \bot, x_2 = \bot, y_1 = \bot$$

$$x_1 = \top, x_2 = \top, y_1 = \top$$

...

2QBF

- Universal (∀) and Existential (∃) Quantifiers

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$$\forall x_1, x_2 : \exists y_1 : (x_1 \vee \neg y_1) \wedge (x_2 \vee \neg y_1)$$

2QBF

- Universal ($\forall$) and Existential ($\exists$) Quantifiers

$$\forall x_1, x_2 : \exists y_1 : (x_1 \vee \neg y_1) \wedge (x_2 \vee \neg y_1)$$

- Two-Player Game between $\forall$ and $\exists$

2QBF

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$$\forall x_1, x_2 : \exists y_1 : (x_1 \vee \neg y_1) \wedge (x_2 \vee \neg y_1)$$

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- Result: true or false
- Models: **Skolem Functions** (ψ)
  - Winning Strategies for $\exists$-Player

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- Two-Player Game between ∀ and ∃
- Result: true or false
- Models: **Skolem Functions** (ψ)
  - Winning Strategies for ∃-Player
  - ∀-variables as inputs
  - ∃-variables as outputs

True 2QBF

Example

$$\forall x_1, x_2 : \exists y_1 : (x_1 \lor \neg y_1) \land (x_2 \lor \neg y_1)$$

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$$\forall x_1, x_2 : \exists y_1 : (x_1 \vee \neg y_1) \wedge (x_2 \vee \neg y_1)$$

x_1	x_2	y_1

True 2QBF

Example

$$\forall x_1, x_2 : \exists y_1 : (x_1 \vee \neg y_1) \wedge (x_2 \vee \neg y_1)$$

x_1	x_2	y_1
0	0	0

True 2QBF

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$$\forall x_1, x_2 : \exists y_1 : (x_1 \vee \neg y_1) \wedge (x_2 \vee \neg y_1)$$

x_1	x_2	y_1
0	0	0
0	1	0
1	0	0

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0	0	0
0	1	0
1	0	0
1	1	0/1

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$$\forall x_1, x_2 : \exists y_1 : (x_1 \lor \neg y_1) \land (x_2 \lor \neg y_1)$$

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0	0	0
0	1	0
1	0	0
1	1	0/1

$$y_1 = \psi_1(x_1, x_2)$$

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0	0	0
0	1	0
1	0	0
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$$y_1 = \psi_1(x_1, x_2) = \begin{cases} \bot \\ \end{cases}$$

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0	0	0
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False 2QBF

Example

$$\forall x_1, x_2 : \exists y_1 : (x_1 \vee \neg y_1) \wedge (x_2 \vee \neg y_1) \wedge (\neg x_1 \vee \neg x_2)$$

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x_1	x_2	y_1
0	0	0
0	1	0
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False 2QBF

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x_1	x_2	y_1
0	0	0
0	1	0
1	0	0
1	1	?

We cannot satisfy this case with y_1.

False 2QBF

Example

$$\forall x_1, x_2 : \exists y_1 : (x_1 \lor \neg y_1) \land (x_2 \lor \neg y_1) \land (\neg x_1 \lor \neg x_2)$$

x_1	x_2	y_1
0	0	0
0	1	0
1	0	0
1	1	**?**

We cannot satisfy this case with y_1.

There is no Skolem Function ...

False 2QBF

Example

$$\forall x_1, x_2 : \exists y_1 : (x_1 \vee \neg y_1) \wedge (x_2 \vee \neg y_1) \wedge (\neg x_1 \vee \neg x_2)$$

x_1	x_2	y_1
0	0	0
0	1	0
1	0	0
1	1	¯_(ツ)_/¯

We cannot satisfy this case with y_1.

There is no Skolem Function
- let's change the definition!

Redefinition: Skolem Function

- Assignments to $\forall$-variables $\rightarrow$ SAT Subformulas

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If X has no non-trivial factors, there is no conventional Skolem Function for Y_1, Y_2.
But with the Redefinition we can model it.

$$\forall X : \underbrace{(\exists Y_1, Y_2 : G(X, Y_1, Y_2))}_{X \text{ has non-trivial factors}} \Leftrightarrow \underbrace{G(X, \{Y_1, Y_2\} = \psi(X))}_{\psi \text{ returns non-trivial factors of } X}$$

Application

- Synthesis of Circuits with unknown Don’t-Cares

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- (potentially) Synthesis of larger Circuits

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- (potentially) Synthesis of larger Circuits
- Information about false QBFs

Future Work

- Survey on Algorithms & Tools for Skolem Function Synthesis (nearing submission)

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- Pre- and Postprocessing of synthesized Skolem Functions (early stage)

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Extra: Origin of Skolem Function Definitions

- Original (uninterpreted Functions in First Order Logic)
  Th. Skolem : ”Logisch-kombinatorische Untersuchungen über die Erfüllbarkeit oder Beweisbarkeit mathematischer Sätze nebst einem Theorem über dichte Mengen”, 1920
- Standard (only for true 2QBFs)
  M. Benedetti : ”sKizzo: A Suite to Evaluate and Certify QBFs”, 2005
  http://link.springer.com/10.1007/11532231_27
- Redefinition (for all 2QBFs)
  S. Akshay et al.: ”What’s hard about Boolean Functional Synthesis?”, 2018
  http://arxiv.org/abs/1804.05507